CYCLICAL NATURE OF PROBLEM-SOLVING PROCESS: CASE STUDY OF TRAINEES IN A TEACHERS' TRAINING INSTITUTE

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This study explores the mathematical problem-solving process of trainee teachers in a teachers' training institute. This research adopts a constructivist perspective that views learning mathematics as a process of constructing meaningful representation and knowledge as being constructed by the individual. The research methodology employs the case study using interpretative approach. Three trainees majoring in mathematics from the Post Graduate in Teaching Course were selected by purposive sampling. Data were collected by audio and video taping the trainee teachers while they were solving three tasks individually using the 'think-aloud' technique, followed by retrospection and clinical interviews. The findings revealed that the problem-solving process of the trainee teachers did not progress in a linear manner but rather moved back and forth with the earlier phases of problem solving in a cyclical nature.

KEY WORDS: constructivism, mental images, problem solving, metacognition.

Introduction

Problem solving has been an important issue in mathematics education during the last three decades. In recent years, considerable attention has been given to problem solving in mathematics and on ways to help students become better problem solvers. Problem solving is generally recognised as one of the most important components of mathematics (Williams, 2003). It has been stated in the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) that problem solving is an integral part of all mathematics learning. At the Ninth International Congress of Mathematics Education, ICME-9 (Pehkonen, 2000), it was highlighted that increasing emphasis has been given to problem solving in the teaching of school mathematics. In addition, the rise of constructivist approaches in learning has further increased its importance in mathematics education. Studies reported in the *Professional Standards for Teaching Mathematics* (National Council of Teachers of Mathematics, 1991) indicated that there was a shift of teaching and learning of mathematics to the constructivist approach. Moreover, problem solving was still being regarded as an act of mental discipline of sense making in order that mathematics can be used in a meaningful way (Polya, 1957; Schoenfeld, 1992). However, comparatively limited studies have been done to investigate the problem-solving abilities and thought processes of students in developing countries such as Malaysia (Lee, 2002).

Literature Review

Beginning in the 1980s until the end of the 1990s, research in mathematical problem solving centred on the role of metacognition and was especially concerned with knowledge of one's own thought processes, and regulation and monitoring of one's activity during problem solving. Among the early references on the role of metacognition as a driving force in mathematical problem solving were studies done by Silver (1982) and Schoenfeld (1982). By the end of 1980s, metacogniton had been linked to various non-cognitive factors; specifically in areas related to beliefs and attitudes (Schoenfeld, 1987; Lester, Garofalo, & Kroll, 1989). However, the degree to which metacognition has influenced problem-solving activity has not been fully understood and indeed, mathematics educators were only just beginning to understand the relationships involved with problem solving (Lester, 1994).

According to Chinnappan (1998), research on mathematical problem solving at that time was focused on issues concerning students' ability at accessing and making flexible use of their previously learnt knowledge. At the Ninth International Congress of Mathematics Education held in Tokyo (Pehkonen, 2000) a special topic study group was set up for the Congress in line with the aim to provide a platform for researchers to share their ideas on the state-of-art in problem solving from various parts of the world. Based on the reports of this topic study group, an important emphasis was the narrowing of focus from group observation to the individual for in-depth exploration of problem-solving behaviours. Concurrent with the increased importance of problem solving in classroom instruction was also the growing emphasis on constructivist views towards the learning and teaching of problem solving. Pehkonen (2000) added a new perspective of employing the knowledge of constructivism to acquire a better understanding of problem-solving processes. The study group also revealed that the constructivist theoretical framework had been employed in studies relating to problem solving. Clinical methods had been used to examine individuals working at problem solving tasks.

In order to enhance students' mathematical learning, teachers should select problems that require the use of problem solving and mathematical thinking in order to give students the opportunity to analyse and explore alternative methods presented. When students learn standard algorithms by rote learning, they often come to think of this as one way among many. Thus, asking students to analyse alternative methods and suggest their own methods gives them the opportunity to make more sense of mathematical learning. Ellis and Yeh (2008) found students who analysed alternative methods and invented their own methods were truly doing mathematics, solving problems and using reasoning in understanding mathematics. As such, exposing students to real-life situations in problem solving allows the students the opportunity to be always searching for efficient, useful and alternative methods. Shore and Pascal (2008) further revealed that pictorial representation was a powerful thinking tool for students who had purposefully used drawings to reason about the situation. Subsequently several distinctly different solution paths emerged in the course of solving the problem. By using pictorial representations, students may also find an unexpected connection, which in turn may provide the opportunity for an informal and healthy discussion of interesting concepts outside the intended curriculum. According to Woodward (2000), better problem solvers were willing to spend more time generating solutions, and they tend to generate a greater variety of possible solutions.

Although traditional step-by-step algorithms used in problem solving may be efficient, they do not allow students to think and see why the methods work. Research studies conducted by Zevenbergen (2005) on pre-service teachers' understanding of volume indicated that some pre-service teachers developed insights into working through the problems, while others relied on algorithmic approaches in their existing schemas to work mathematically. The study revealed that there was a heavy reliance on procedural knowledge and algorithmic methods in which lock-step strategies were used to solve the mathematical task. Scenarios where the teacher gives students a step-by-step procedure for solving the problem or leads them through a solution in the form of an activity sheet defeats the purpose of engaging them to think about and represent mathematical concepts (Rabin, 2008).

In a Malaysian study, Koay (2007) found that the mathematical problem solution paths adopted by the trainee teachers were related to their mental images of the problem task. In this sense, it was observed that in the planning phase of the problem solving process, the trainee teachers were often engaged in a sub-cycle "*conjecture-mental image-evaluate*" that appear as mental images during which the trainee teachers try to imagine and construct various solution approaches, conjecturing and evaluating the viability of the conjectured approach. The finding supports Carlson and Bloom's (2005) multidimensional problem solving framework.

It is therefore seen that the recent emphasis in problem solving research is more focused on knowledge construction, the understanding of problem solving schemes and mental structures in mathematical problem solving. However, research conducted on trainee teachers is rather limited, thus the findings concerning trainee teachers' mathematical problem solution strategies and thought processes in mathematical problem solving aims to contribute further towards enriching the knowledge of mathematics educators regarding this area of study

Purpose of the Study

The Research Questions

This study attempts to explore the mental images of the trainee teachers on problem solving and their understanding of problem solving. Secondly, this study intends to examine the mathematical problem solution path and thought processes of the problem solvers while attempting to solve mathematical problems. This study therefore attempts to answer the following research questions:

1. What were the mental images of trainee teachers as they attempted to solve the mathematical tasks?
2. What were the mathematical problem solution paths used by the trainee teachers to solve the given mathematical tasks?

The Conceptual Framework of the Study

The conceptualisation of this framework is based on the constructivist perspective (see Figure 1). When the trainee teacher is given a problem solving task, it provides a situation that triggers an action or operation. The trainee teacher constructs an initial representation or mental image while reading the task. Mental images emerge as the trainee teacher tries to understand and make sense of the task situation. The trainee teacher's prior knowledge and mathematical learning experience in their schooling constitutes the trainee teacher's mathematical knowledge. As the trainee teacher attempts to solve the task, assimilation and accommodation take place through the process of reflective abstraction. The process of reflective abstraction that is also the instrument of accommodation results in actions or operations. Finally, there is a result or a sequel of activity that manifests in the problem solving. The coordination and reflection on the interaction with the task situation result in the mathematical problem solving schemes. These interactions between the task and the internalised mathematical problem solving schemes are critical to the solution of a problem. Thus, 'the use of schemes' by the trainee teacher results in the development of the solution path to which the scheme is applied when the trainee teacher attempts to solve the task. The task is successfully solved when the trainee teacher is able to assimilate and accommodate it through reflective abstraction and a more complete adequate solution path is obtained.

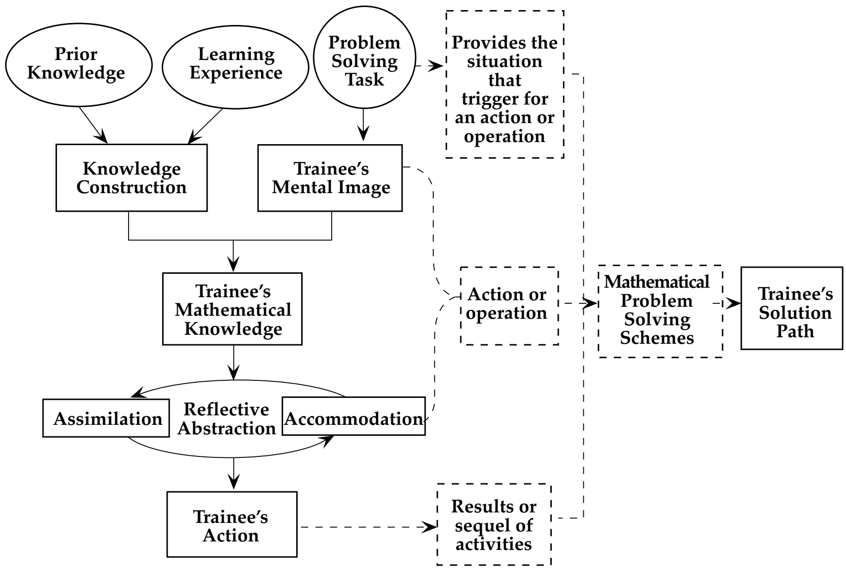


Figure 1. Conceptual framework of the study.

Research Design and Methodology

The research design comprised of three parts which involved a set of three problem solving tasks, using the 'think-aloud' techniques in the problem solving process, retrospection schedule, and the clinical interview session.

In the first part of the data collection, the subjects were required to use the 'think-aloud' technique while solving the problem solving tasks. Prior to the problem solving session, the subjects were given a brief training session regarding the 'think-aloud' technique. During the 'think-aloud' problem solving session, the researcher observed the subjects' behaviours and also took notes. When necessary, the researcher would periodically remind the subjects to verbalise their thinking and articulate a rationale for specific behaviours. During the problem-solving session, it was important for the researcher to remain non-directive until the subject arrived at a solution that he or she felt was acceptable. It was only during the clinical interviews that probes were made with regard to the subjects' thought processes and problem solving behaviours so as to determine the underlying basis of the subject's responses and solutions path.

The retrospection session which formed the second part of the data collection was conducted immediately after the problem-solving session. The subjects were requested to write a reflection on their thought processes and to explain why they made such an attempt in the solution path. The aim of the retrospection was to infer the subject's thought processes and to put their 'thinking during problem solving' in written form.

The third part of the data collection was the clinical interviews which were conducted a few days after the problem solving and retrospection sessions. This was to allow the researcher some time to analyse the data collected from the problem solving tasks, 'think-aloud' protocol and retrospection sessions. The interview session was intended to explore further the reasons for the respondent's particular actions in their solution path during the problem- solving session.

The Respondents

The three respondents were purposively selected from a group of trainee teachers majoring in mathematics in the Post Graduate Diploma in Teaching Course from a teacher training institute. They were in the second semester in the course for primary school education at the time the data were collected. All the trainee teachers in the course have a Bachelor Degree from various disciplines.

Findings and Discussion

This section attempts to describe the problem solving process and behaviours exhibited by three teacher trainees: Penny, Mat and Kofi, when attempting to solve the three tasks.

Task 1

En. Ariffin left his house to drive to his office in Subang Jaya. His car's odometer read 25678 km. He drove 5 km but realised that he had left his files at home. So, he returned home, picked up his files, and drove to his office. After work, he turned around and drove directly home. His odometer now read 25786 km. How far is En. Ariffin's home from his office in Subang Jaya?

Penny's case in solving Task 1. Penny was able to solve Task 1 rather quickly with the help of the drawing as it gave a clearer picture of the whole journey taken by En. Ariffin (See Table 1). She also wrote down the car's odometer reading as well as the distances travelled from his house to the office and then back to his house after working on the diagram. This ensured that she did not miss out on any particular points as given in the task. Excerpts from Penny's interview session were as follows:

Researcher: *Did you imagine anything when you were constructing the diagram?*

Penny: *Yes, I imagine.*

Researcher: *What did you imagine?*

Penny: *I imagine a car with a house and an office.*

Researcher: *So, this diagram... did it help you to answer this problem?*

Penny: *Yes, it helped.*

Researcher: *How did it help?*

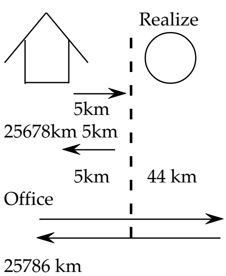
Penny: *Because from this diagram, I could clearly draw how far the distance from his house to the place where he realised he has forgotten his files. Then, he returned from his house to the office and from the office to his house. It could be seen clearly as I will not 'miss out' any of them.*

Penny's mental image:

It appears to me a picture when I read this task. I thought that I wanted to draw out so that it is easier for me to solve this problem.

In the retrospection, Penny expressed that *"By using diagram it was easier for me to find the required distance. This task was rather easy to understand. It builds up my confidence to continue attempting the tasks subsequently."* The transcripts of Penny's 'think-aloud' protocol and analysis for Task 1 are displayed in Table 1.

Table 1
Analysis of Penny's Mathematical Problem Solution Path and Observed Behaviour for Task 1

Solution path	Transcripts/Explanation	Behaviour	Phase
Code: TABP2/2 <u>Basic operations split into 2 and drawing scheme</u>	Reads task. Reads task again and as she read, she drew the diagram to depict the journey taken by En. Ariffin.	Initial contact Sense making Conjecture-mental image-evaluate	Understanding Understanding Planning
	*(BP2.1) ……After work, he turned around and drove directly home. He returned home and now odometer showed 25786 km. Here it means 5 km, 5 km, 5 km.* (She wrote 5 km 3 times on the diagram.)	Reflecting and drawing	Planning
	(BP2.2) Ok, the total distance travelled equals 25786 - 25678. Means that total distance travelled was 108 km.	Computing	Carrying out
Total distance travelled $\quad 25786$ km $- \quad$ <u>25678 km</u> $\quad$ <u>108 km</u>	(BP2.3) If we know that when he realised that he had forgotten his files was 5 km.	Reflecting	Managerial process
Total distance 108 km, in his journey there were four 5 km.	(BP2.4) Ok, so there are 1, 2, 3, 4, four 5's.There are four distance of 5 km in the journey.	Knowledge construction	
Thus, 108 km - 20 km = 88 km	Thus, 108 km - 20 km equals 88 km.	Computing	Carrying out
88 km ÷ 2 = 44 km	(BP2.5) 88 km ÷ 2 equals 44 km because to and fro, 44km, 44 km.	Computing	Carrying out
The distance of En. Ariffin's home from his office in Subang Jaya is 44 km + 5 km = 49 km.	(BP2.6) Therefore, the distance of En. Ariffin's house from his office in Subang Jaya is 44 km + 5km equals 49 km.	Getting a solution	Managerial decision
Checking back: 49 km + 49 km + 10 km 108 km	(BP2.7) Checking back 49 km + 49 km + another 10 km gives 108 km.	Verifying	Checking back Completion
Answer: 49 km (correct)			

Task 2

John has a machine that operates only on the given number and no other. Thus, if he inputs 3, the machine can only operate on 3's. The machine uses the four basic operations of arithmetic (addition, subtraction, multiplication, and division) either alone or in combination. Here are the first six outputs for inputs of $x = 1$ to 6:

Input (x)	*Output*
1	*1*
2	*9*
3	*29*
4	*67*
5	*129*
6	*221*

What is the value of output, if we input 9?

Mat's case in solving Task 2. Mat's initial concern was to read and to understand the task. He did not understand the statement *'if he inputs 3, the machine can only operate on 3's* (Table 1.2, BS3.2). Initially, he thought that the task involved progression and after reading three times he tried to find a relationship between the inputs and outputs. Excerpts from Mat's interview session were as follows:

> *Ah, this task, at first I feel it was progression. Previously, progression is definitely a topic that I was not interested. Then I left aside after reading it because I thought it was progression that I have no interest. I left for the other tasks but last option I still need to do it. So, I went through two or three times after that I tried to do. I used trial and error method to find the relationship between the inputs and outputs. Ah, after I obtained the relationship I tried to find what is required.*

Mat's mental image:

Initially, I thought it was progression then later I tried using trial and error.

Mat tried out with input 2 and the next few inputs until input 6 (BS3.4, BS3.6, BS3.7). After that he tried testing with input 7, 8 and 9 and finally, he got a relationship $9^3 + 8 = 737$ (BS3.10, BS3.11, BS3,12). The transcripts of Mat's 'think-aloud' protocol and analysis for Task 2 are displayed in Table 2.

Table 2

Analysis of Mat's Mathematical Problem Solution Path And Observed Behaviour for Task 2

Solution path	Transcript/Explanation	Behaviour	Phase
Code: BS3/4	Reads task. (BS3.1)I will try to find a relationship between each input and output (pause).	Initial contact Conjecture-mental image-evaluate	Understanding Planning
Patterns in numbers using basic operations 1 → 1 2 → 2 X 2 X 2 + 1 3 → 3 X 3 X 3 + 2	(BS3.2) Ok, before that I do not understand the statement 'If he input 3, the machine can only operate on 3's.'	Reflecting on the problem statement	Managerial process
4 → 4 X 4 X 4 + 3 = 67 5 → 5 X 5 X 5 + 4 = 129 6 → 6 X 6 X 6 + 5 = 221 7 → 7 X 7 X 7 + 6	(BS3.3) 'Operates on 3's', probably number in multiples of 3, probably also number in multiples of 3 such as prime numbers.	Conjectures	Planning
8 → 8 X 8 X 8 + 7 9 → 9 X 9 X 9 + 8 = 737	(BS3.4) I tried to find relations of 2, 4 (pause).	Strategizing – finding a relation-ship	Planning
Answer: 737 (correct)	(BS3.5) Is the product of output more than the *operation that involves number of input only? (pause)*	Reflecting on the input and output	Managerial process
	(BS3.6) For example, input 2 produced output 9.	Sense making	Understanding
	(BS3.7) Is this output 9 a product from the combination or operation +, -, X, division of number 2 only. That is playing in my mind at this instance	Conjecture-mental image -evaluate	Planning

Solution path	Transcript/Explanation	Behaviour	Phase
	(BS3.8) 'Operates on 3's'.	Sense making	Understanding
	(BS3.9) Ok, tried get a relation for these numbers.	Reflecting	Managerial process
	(BS3.10) Ok, now I will try to obtain a relationship for input 9 (pause).	Finding a relationship.	
	(BS3.11) 7 X 7 X 7 + 6; 8 X 8 X 8 + 7; 9 X 9 X 9 + 8 .	Computing	Carrying out
	(BS3.12) Ok, so (pause) I feel the number of output for input 9 is 737.	Getting a solution	Managerial decision
	(BS3.13) This is based from the relationship that I obtained from the given information ah that is 93 + 8.	Reflection	Managerial process Completion

Task 3

A local pet shop owner just bought her holiday supply of baby chickens and baby rabbits. She bought a total of 22 animals which had a total of 56 legs. How many chickens and how many rabbits did she buy?

Kofi's case of solving Task 3. Kofi's initial contact was to read the task and to understand the problem statement. He was contemplating using equations to solve this task (see Table 3, BS4.1). He let the variables be M for number of chickens and B for number of rabbits (BS4.4). He was able to construct the algebraic relationship with the number of chickens and rabbits to make a total of 22; and also the legs of chickens and rabbits to make a total of 56 by forming two linear equations with two variables (BS4.5). Then using simultaneous equations, he solved and obtained B as 6 and M as 16 (BS4.7, BS4.8, BS4.9, BS4.10, BS4.11). Kofi checked the solutions by substituting the solutions for B and M into the two linear equations to obtain the total number of animals and the total number of legs for both chickens and rabbits (BS4.12, BS4.13, BS4.14). He was quite confident in getting the correct solutions (BS4.15). The transcripts of Kofi's 'think-aloud' protocol are shown in Table 3.

Table 3
Analysis of Kofi's Mathematical Problem Solution Path and Observed Behaviour for Task 3

Solution path	Transcripts/Explanation	Behaviour	Phase
Code: TABS4/3 Using simultaneous equation (M) Chicken = 2 legs	Reads task. (BS4.1) Ok, looks like the trend of this task is also equation.	Initial contact Conjecture-mental image-evaluate	Understanding Planning
(B) Rabbit = 4 legs Form two linear equations. M + B = 22 …. (1)	(BS4.2) Because it says number of legs, we need to first ensure chicken has how many legs.	Strategizing	Planning
2M + 4B = 56 …. (2) (1) X 2, 2M + 2B = 44 …… (3)	(BS4.3) Ok, normally chicken has 2 legs, rabbit has 4 legs. Total given was 22 animals.	Reflecting	Managerial process
(2) - (3), 2B = 12 B = $\frac{12}{2}$	(BS4.4) Ok, now we assume chickens represented by unknown M, rabbits represented by unknown B.	Conjecture	Planning
= 6	(BS4.5) Hence, equation (1) is M + B = 22; 2M + 4B = 56; equation (2).	Knowledge construction	
Substitute B = 6 in equation (1) M + 6 = 22 M = 22 - 6 = 16	(BS4.6) Now, we get two equations with 2 unknowns. (BS4.7) We can solve using elimination method.	Reflecting on the equations Strategizing	Managerial process Planning
Number of chickens = 16 (correct)	(BS4.8) First, equation (1) multiply by 2. We will get 2M + 2B = 44 equation (3).	Computing	Carrying out
Number of rabbits = 6 (correct) Checking: Number of animals: 6 + 16 = 22	(BS4.9) Ok, to eliminate M, we need to subtract equation (2) from equation (3). …. Hence, B is $\frac{12}{2}$ = 6.	Computing	Carrying out
Number of legs: 6(4) + 16(2) = 24 + 32 = 56	(BS4.10) Ok, substitute B = 6 in equation (1), M + 6 = 22. Hence, M = 22- 6, we will get 16.	Computing	Carrying out
	(BS4.11) Ok, so number of number of chickens represented by M is 16. Number of rabbits represented by B is 6.	Getting a solution	Managerial decision

Solution path	Transcripts/Explanation	Behaviour	Phase
	(BS4.12) Ok, to perform checking back. Checking back we put the values of number of chickens and number of rabbits into the previous equation.	Verifying	Checking back
	(BS4.13) So, number of rabbits 6 add number of chickens 16 equals 22.	Verifying	
	(BS4.14) Ok, number of legs (pause).) 6 rabbits, number of legs 4 legs; 16 chickens, number of legs 2 equals to 24 add 32 get 56 legs.	Verifying	
	(BS4.15) So, the solutions I obtained are surely correct.	Confirming the solution	Completion

Kofi was able to comprehend and solve Task 3 with ease. He was able to represent the relationships of the data by constructing two algebraic equations with two unknowns. Kofi showed that he possessed a strong fundamental in algebra through the systematic presentation of his written work as shown in Table 1.3. In the retrospection, he wrote as follows:

This task is slightly more difficult than Task 2 but easier than Task 4. This task made me recalled the mistakes that I usually made in school. However, now I understand the requirement of the task and was able to solve it. The most important is that we have to construct a few equations from the given problem.

Conclusion

Based on the analysis of the thought processes and observed behaviours of the subjects during the problem-solving session, it was observed that the majority of the subjects had followed through the four phases of problem solving namely understanding, planning, carrying out, and checking. However, the subjects' thought processes did not progress in a linear manner but instead moved back and forth with the earlier phases of problem solving. The dotted arrows in the diagram indicate that after the planning phase, the solver engaged in the process of knowledge construction before moving on to the next phase that is carrying out the planned activity. In the process of knowledge construction, reflective abstraction took place and the solver engaged in the process of assimilation and accommodation.

Thus, the analysis of the subjects' 'think-aloud' problem solving sessions indicate that the problem solving process is cyclical in nature. It was interesting to note that during the initial stage of the planning phase, when the subjects were engaged in sense making and contemplating on the various problem solving approaches, the subjects were at times engaged in a sub-cycle comprising of 'conjecture – mental image – evaluate' (see Figure 2). This normally appeared in the subjects' thought processes as mental images rather than in written form when the solver was considering the viability of various strategies adopted.

It was also noted that the subjects also tended to engage in cognitive and metacognitive processes in their problem-solving attempts. The cognitive processes involved logical reasoning, knowledge construction and making sensible conceptual connections. Metacognitive processes involved self-monitoring and reflecting on the efficiency and effectiveness of their cognitive activities and solution attempts. The metacognitve behaviours of the subjects during their problem solving processes are referred to as the managerial processes in the problem solving framework.

During the understanding phase, the subjects would be reading the task, trying to make sense of the problem statement, and organising the facts. As the subjects attempt to make sense of the task, they would normally be engaged in constructing mental images to represent the task situation. In the planning phase, the subjects would be constructing conjectures, defining unknowns with variables, constructing hypothetical prepositions, or conjecturing about a viable solution approach. The subjects while contemplating on the various strategies or solution approaches by imagining the task situation while considering the viability of possible strategies.

In the carrying out phase, the subjects normally engaged in behaviours involving knowledge construction, conducting computations, drawing, writing out logically connected mathematical statements, executing strategies or procedures, and carrying out the heuristic of problem solving. During the checking out phase, the subjects attempted to verify the solution path by assessing the correctness of their computations and results, and going through the heuristic or procedures of their solution path.

Thus, it was noted that the subjects rarely solved a task by working in a linear manner but rather the processes tend to be cyclical in nature. The successful solvers sometimes made multiple attempts and go back and forth and cycled through the various phases of the problem solving framework. Sometimes this cycle appeared to be slow and tedious, while at times it may appear to be fast when the solver managed to strike at the right chord and make the correct connection. The findings of this study revealed that the subjects' problem solving process concurred with Wilson, Fernandez and Hadaway's (1993) dynamic and cyclical problem solving model and the sub-cycle of Carlson and Bloom's (2005) multidimensional problem-solving framework.

In the process of solving the task, the subjects regularly engaged in metacognitive behaviours that involved reflecting on the effectiveness and efficiency of the solution path or reflecting on one's problem-solving process. These acts of reflections and self-monitoring behaviours were more predominant among successful solvers. In conclusion, the researcher proposes a modified emergent problem solving framework with the inclusion of two important components, which is the sub-cycle 'conjecture – mental image –

evaluate' and knowledge construction to the Wilson's dynamic and cyclical problem solving model as depicted in Figure 2.

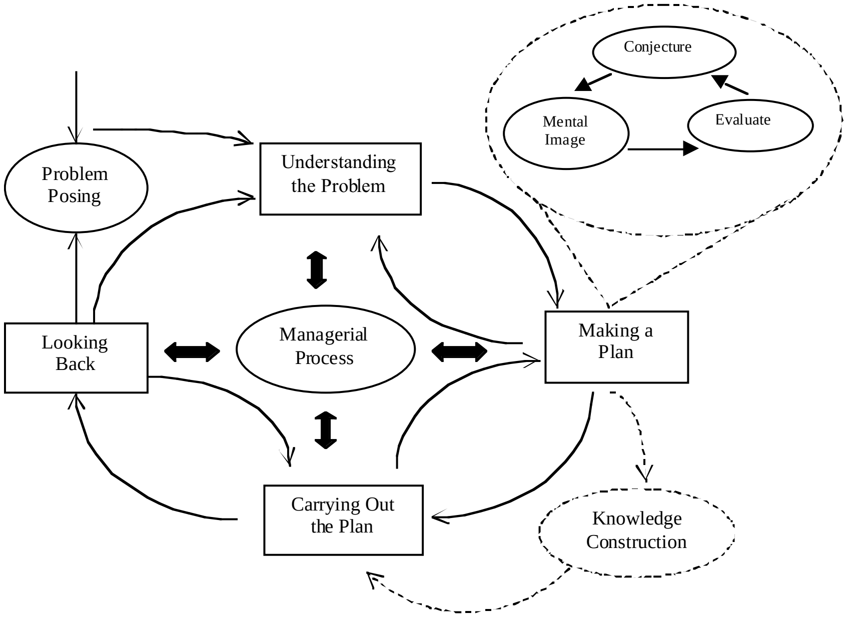


Figure 2. Modified emergent dynamic and cyclical nature of problem solving framework.

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